Graphs with Extremal Incidence Energy

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Abstract. For a simple connected graph G, the incidence energy $IE(G)$ is defined as the sum of all singular values of its incidence matrix. In this paper, we characterize the graphs with the maximum incidence energies among all graphs with given chromatic number and given pendent vertex number, respectively. We also characterize the graphs with the minimum incidence energy among all graphs with given clique number. Especially, we characterize the tree with the minimum incidence energy among all trees with given pendent vertex number.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A(G) = (a_{ij})$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if $v_i v_j \in E$. The eigenvalues of $A(G)$ are called the eigenvalues of G.

The notion of the energy of a graph was introduced by Gutman as the sum of the absolute values of its eigenvalues, it is studied in chemistry and used to approximate the total-electron energy of a molecule [4]. The singular values of an $n \times m$ matrix M are the nonnegative square roots of the eigenvalues of MM^* if $n \leq m$ or M^*M if $n \geq m$, where M^* is the transpose conjugate of M. Nikiforov [13] extended the concept of energy to all matrices M, defining the energy of a matrix M as the sum of the singular values of M. Clearly, the energy of the matrix $A(G)$ is just the energy of the graph G from the fact $A(G)^* = A(G)$.

Let edges of G be given an arbitrary orientation producing an oriented graph $\vec{G}$, and let $B(G)$ be the vertex-edge incidence matrix of the oriented $\vec{G}$, whose $(v; e)$ entry is equal to +1 if the vertex v is the head of the oriented edge e, −1 if v is the tail of e, and 0 otherwise. The oriented incidence energy of G, denoted by $OIE(G)$, is defined by Stevanović et al. in [15] as the energy of the matrix $B(G)$. This invariant is also called the Laplacian-like energy $LEL(G)$ of a graph G in [9]. Various properties of $OIE(G)$ or $LEL(G)$ were found in [10, 15–19].

Let $X(G)$ be the (vertex- edge) incidence matrix of G with $x_{ij} = 1$ if v_i is incident to e_j, and $x_{ij} = 0$ otherwise. In analogy to the oriented incidence energy, the incidence energy of G is defined as the energy

of the matrix $X(G)$ in [5, 7]. That is, suppose that $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of $X(G)$, the incidence energy of a graph G is defined as

$$IE(G) = \sum_{i=1}^{n} \sigma_i. \tag{1}$$

The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G (see [1, 2]), where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal matrix which entries are the degrees of vertices of G, respectively. Let $q_1, q_2, \ldots, q_n$ be the eigenvalues of $Q(G)$. Note that $X(G)X^T(G) = D(G) + A(G) = Q(G)$, thus the incidence energy of a graph G is also defined as [5]

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i(G)}. \tag{2}$$

For a bipartite graph G, $IE(G) = OIE(G)$, since the signless Laplacian $Q(G)$ and the Laplacian matrix $L(G) = D(G) - A(G) = B(G)B(G)^T$ have common eigenvalues.

Tan and Hou [20] determined the graphs with minimal and maximal incidence energy among the trees on n vertices. Zhang and Li [22] determined the graphs with respectively minimal and maximal incidence energy respectively among unicyclic graphs and bicyclic graphs on n vertices. Some properties of the incidence energy of graphs were obtained in [5, 7], and several upper and lower bounds for the incidence energy of graphs were derived in [6, 23]. Mirzakhah et al. [11] gave many graph transformations on the coefficients of the signless Laplacian characteristic polynomial. In fact, those are also suit-able for the incidence energy of graphs. In [14], a sharp upper bound for the incidence energy of the graphs G on n vertices with vertex connectivity less than or equal to k was obtained.

A tree is said to be starlike if it has exactly one vertex of degree greater than 2. Let $S_{n,k}$ denote the starlike tree on n vertices with k pendent paths of almost equal length and $S_{n,k-1}^a$ be the graph obtained by joining a pendent vertex of path P_{a-1} to the center of $S_{n-a+1,k-1}$. Let $K_{n,s}^{n_1,n_2,\ldots,n_s}$ be the graph obtained from a complete graph K_s by respectively attaching n_i pendent vertices to v_i for $i = 1, 2, \ldots, s$, where $V(K_s) = \{v_1, v_2, \ldots, v_s\}$ and $n_1 + n_2 + \ldots + n_s = n - s$. Let $K_{n,s} = K_{n,s}^{n_1,n_2,\ldots,n_s}$ with $n_1 = n - s, n_2 = n_3 = \cdots = n_s = 0$ and $K_{n,s}^s = K_{n,s}^{n_1,n_2,\ldots,n_s}$ with $\max_{1 \le i,j \le s} |n_i - n_j| \le 1$. Recall that the Turán graph $T_{n,r}$, is a complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets. In this paper, we will prove that the Turán graph $T_{n,\chi}$ is the unique graph with maximum incidence energy in the set of all graphs on n vertices with chromatic number χ, $K_{n,n-k}^{n-k}$ is the graph with maximum incidence energy in the set of all graphs on n vertices with k pendent vertices, respectively. We also will prove that $K_{n,\omega}$ is the unique graph with minimum incidence energy in the set of all graphs on n vertices with clique number ω. Finally, we will prove that $S_{n,k}$ is the unique tree with minimum incidence energy in the set of all trees on n vertices with exactly k pendent vertices.

2. Preliminaries

Let $S(G)$ be the subdivision graph of the graph G, which obtained by inserting an additional vertex into each edge of G. Then $S(G)$ is a bipartite graph. Suppose that the characteristic polynomial of $S(G)$ is [1]

$$P_{S(G)}(x) = det(xI - A(S(G))) = \sum_{i=0}^{\lfloor \frac{n+m}{2} \rfloor} (-1)^i b_{2i}(S(G)) x^{n+m-2i},$$

where $b_{2i}(S(G)) \ge 0$, $b_0(S(G)) = 1$, $b_2(S(G)) = n + m$. Especially, if G is a tree, then $b_{2k}(G) = m(G, k)$, where $m(G, k)$ denotes the number of k-matchings of G. Let the Q-polynomial of G be

$$Q_G(x) = det(xI - Q(G)) = \sum_{i=0}^{n} (-1)^i p_i(G) x^{n-i}.$$

It was proved in [1] that

$$P_{S(G)}(x) = x^{m-n}Q_G(x^2). \tag{3}$$

Hence $b_{2i}(S(G)) = p_i(G)$ for $0 \le i \le n$, and $b_{2i}(S(G)) = 0$ for $n < i \le \lfloor \frac{n+m}{2} \rfloor$.

Let $f(x) = \sum_{i=0}^{n}(-1)^i f_i x^{n-i}$ and $g(x) = \sum_{i=0}^{n}(-1)^i g_i x^{n-i}$ be two polynomial of degree n. We call $f(x)$ majorises $g(x)$, denoted by $f(x) \ge g(x)$ if $f_i \ge g_i \ge 0$ for $0 \le i \le n$; Furthermore, $f(x) > g(x)$ if $f(x) \not\equiv g(x)$, and $f(x) > \mathbf{0}$ if all $f_i \ge 0$ and $f_k > 0$ for some k.

Given two graphs G_1, G_2 on n vertices, it was proved in [22] that

$$p_i(G_1) \le p_i(G_2) \quad \text{for} \quad i = 1, 2, \dots, n \implies IE(G_1) \le IE(G_2) \tag{4}$$

Moreover, if there exists some k such that $p_k(G_1) < p_k(G_2)$, then $IE(G_1) < IE(G_2)$.

For a polynomial $h(x) = \sum_{i=0}^{n} h_i x^{n-i}$. Let $C_{x^k}(h(x))$ be the coefficient of x^k in $h(x)$, i.e., $C_{x^k}(h(x)) = h_{n-k}$ for $0 \le k \le n$.

Lemma 2.1. *Let G_1, G_2 be two graphs on n vertices. If $Q_{G_1}(x) \ge Q_{G_2}(x)$, then $IE(G_1) \ge IE(G_2)$; Furthermore, if $Q_{G_1}(x) > Q_{G_2}(x)$, then $IE(G_1) > IE(G_2)$.*

Proof. Since $Q(G)$ is a semi-definite matrix, $Q_G(x)$ is a polynomial which has a positive leading coefficient and the signs of its coefficients are alternating. Combining the definition of $" \ge "$ and Inq.4, the result follows. $\square$

Lemma 2.2. *Let*

$$A(x) = \sum_{i=0}^{s}(-1)^i a_i x^{s-i}, B(x) = \sum_{i=0}^{t}(-1)^i b_i x^{t-i}$$

and

$$C(x) = \sum_{i=0}^{s}(-1)^i c_i x^{s-i}, D(x) = \sum_{i=0}^{t}(-1)^i d_i x^{t-i}.$$

If $A(x) \ge C(x) > \mathbf{0}$ and $B(x) \ge D(x) > \mathbf{0}$, then $A(x)B(x) \ge C(x)D(x)$. Moreover, if $A(x) > C(x)$ or $B(x) > D(x)$, then $A(x)B(x) > C(x)D(x)$.

Proof. Since $a_i \ge c_i \ge 0$ for $0 \le i \le s$ and $b_i \ge d_i \ge 0$ for $0 \le i \le t$,

$$(-1)^k C_{x^{n-k}}(A(x)B(x)) = \sum_i a_i b_{k-i} \ge \sum_i c_i d_{k-i} = (-1)^k C_{x^{n-k}}(C(x)D(x)),$$

and then the first result follows. Suppose that $A(x) > C(x)$. From the first result we have

$$A(x)B(x) > C(x)B(x) \ge C(x)D(x).$$

Thus the second result follows. $\square$

A spanning subgraph of G whose components are trees or unicyclic graphs is called a TU-subgraph of G. Suppose that a TU-subgraph H of G contains $c(H)$ unicyclic graphs and s trees $T_1, T_2, \dots, T_s$. Then the weight $W(H)$ of H is defined as $W(H) = 4^{c(H)} \Pi_{i=1}^{s}(1 + |E(T_i)|)$. Clearly, the isolated vertices in H do not contribute to $W(H)$. D. Cveković etc [2] proved that

$$p_i(G) = \sum_{H_i} W(H_i),$$

where the summation runs over all TU-subgraphs H_i of G with i edges. Thus the following Lemma is obvious.

Lemma 2.3. *[11] Let G be a simple connected graph and $e \in E(G)$, then $p_i(G) \ge p_i(G - e)$ with equality if and only if $i = 0$. That is, $Q_G(x) > Q_{G-e}(x)$, and then $IE(G) > IE(G - e)$.*

3. The Incidence Energy of Connected Graphs with Given Chromatic Number

Let $T_{n,\chi}$ denote the Turán graph. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors assigned to the vertices of G such that adjacent vertices have different colors.

Theorem 3.1. *Let $\mathbb{G}_{n,\chi}$ be the set of all simple graphs on n vertices with chromatic number χ and G be an arbitrary graph in $\mathbb{G}_{n,\chi}$. Then*

$$IE(G) \leq IE(T_{n,\chi})$$

with equality if and only if $G \cong T_{n,\chi}$.

Proof. Let G be a graph which has the maximal incidence energy in $\mathbb{G}_{n,\chi}$. The result is trivial for $\chi = 1$. Now we suppose that $\chi \geq 2$. By the definition of chromatic number, we have that each color class of G is an independent set. Let $V^1, V^2, \ldots, V^\chi$ be the χ-color classes of G where each V^i is an independent set with $|V^i| = n_i$. By Lemma 2.3, $G \cong K_{n_1,n_2,\ldots,n_\chi}$. Let $n_1 = \max_{1 \leq i \leq \chi} n_i$.

Suppose that $G \ncong T_{n,\chi}$. Then there exists i such that $n_1 - n_i \geq 2$. Without loss of generality, we suppose that $n_1 - n_2 \geq 2$. Then let $G_1 \cong K_{n_1-1,n_2+1,\ldots,n_\chi}$. Obviously, $G_1 \in \mathbb{G}_{n,\chi}$. By Theorem 1 of [21], we know that

$$Q_{K_{n_1,n_2,\ldots,n_\chi}}(x) = \prod_{i=1}^{\chi} (x - (n - n_i))^{n_i-1} \left(\prod_{i=1}^{\chi} (x - (n - 2n_i)) - \sum_{i=1}^{\chi} n_i \prod_{\substack{j=1 \\ j \neq i}}^{\chi} (x - (n - 2n_j)) \right).$$

Let

$$f_{n_1,n_2}(x) = \prod_{i=1}^{\chi} (x - (n - n_i))^{n_i-1}$$

and

$$g_{n_1,n_2}(x) = \prod_{i=1}^{\chi} (x - (n - 2n_i)) - \sum_{i=1}^{\chi} n_i \prod_{\substack{j=1 \\ j \neq i}}^{\chi} (x - (n - 2n_j)).$$

Then $Q_G(x) = f_{n_1,n_2}(x) g_{n_1,n_2}(x)$ and $Q_{G_1}(x) = f_{n_1-1,n_2+1}(x) g_{n_1-1,n_2+1}(x)$.

Obviously, $x - (n - n_2 - 1) > x - (n - n_1)$ and $x - (n - n_1 + 1) > x - (n - n_1)$. By Lemma 2.2, we have

$$(x - (n - n_2 - 1))^{n_1-n_2-1} (x - (n - n_1 + 1)) > (x - (n - n_1))^{n_1-n_2}.$$

Similarly, by the fact

$$(x - (n - n_1 + 1))(x - (n - n_2 - 1)) > (x - (n - n_1))(x - (n - n_2))$$

and Lemma 2.2, we have that

$$[(x - (n - n_1 + 1))(x - (n - n_2 - 1))]^{n_2-1} > [(x - (n - n_1))(x - (n - n_2))]^{n_2-1}.$$

Thus $f_{n_1-1,n_2+1}(x) > f_{n_1,n_2}(x)$.

Now considering the difference of $g_{n_1-1,n_2+1}(x)$ and $g_{n_1,n_2}(x)$, we have

$$g_{n_1-1,n_2+1}(x) - g_{n_1,n_2}(x)$$

$$= \quad [(x - (n - 2(n_1 - 1)))(x - (n - 2(n_2 + 1))) - (x - (n - 2n_1))(x - (n - 2n_2))] \cdot \prod_{i=3}^{\chi} (x - (n - 2n_i))$$

$$\quad -[(n_1 - 1)(x - (n - 2(n_2 + 1))) + (n_2 + 1)(x - (n - 2(n_1 - 1)))$$

$$-n_1(x-(n-2n_2)) - n_2(x-(n-2n_1))] \cdot \prod_{i=3}^{\chi}(x-(n-2n_i))$$

$$-[(x-(n-2(n_1-1)))\,(x-(n-2(n_2+1))) - (x-(n-2n_1)))\,(x-(n-2n_2))] \cdot \sum_{i=3}^{\chi} n_i \prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j))$$

$$= \quad 4(n_1-n_2-1)\prod_{i=3}^{\chi}(x-(n-2n_i)) - 4(n_1-n_2-1)\prod_{i=3}^{\chi}(x-(n-2n_i)) - 4(n_1-n_2-1)\sum_{i=3}^{\chi} n_i \prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j))$$

$$= \quad 0 \cdot x^{\chi} - 0 \cdot x^{\chi-1} + 0 \cdot x^{\chi-2} - 4(n_1-n_2-1)\sum_{i=3}^{\chi} n_i \prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j)).$$

If $\chi = 2$, then $g_{n_1-1,n_2+1}(x) = g_{n_1,n_2}(x)$. Suppose now that $\chi \geq 3$. Since $n-2n_j > 0$ for $i = 3,4,\ldots,\chi$, $n_i\prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j))$ is a polynomial of degree $n-3$ which has a positive leading coefficient and the signs of its coefficients are alternating. Then $\sum_{i=3}^{\chi} n_i \prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j))$ is a polynomial of degree $n-3$ which has a positive leading coefficient, and the signs of its coefficients are alternating. Hence for $0 \leq k \leq \chi - 3$,

$$(-1)^k C_{x^{\chi-3-k}}\left(\sum_{i=3}^{\chi} n_i \prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j))\right) > 0.$$

Note that $n_1 - n_2 - 1 > 0$. Then for $0 \leq k \leq \chi - 3$

$$(-1)^k C_{x^{\chi-3-k}}\left(-4(n_1-n_2-1)\sum_{i=3}^{\chi} n_i \prod_{\substack{j=3\ j\neq i}}^{\chi}(x-(n-2n_j))\right) < 0,$$

which implying that $(-1)^k C_{x^{\chi-k}}\left(g_{n_1-1,n_2+1}(x) - g_{n_1,n_2}(x)\right) = 0$ for $k = 0,1,2$, and

$$(-1)^k C_{x^{\chi-k}}\left(g_{n_1-1,n_2+1}(x) - g_{n_1,n_2}(x)\right) > 0$$

for $3 \leq k \leq \chi$. Thus $g_{n_1-1,n_2+1}(x) > g_{n_1,n_2}(x)$ for $\chi \geq 3$. By Lemma 2.2, $Q_{G_1}(x) > Q_G(x)$ and then $IE(G_1) > IE(G)$, a contradiction. Thus $G \cong T_{n,\chi}$. □

4. The Incidence Energy of Connected Graphs with Given Clique Number

By Sachs Theorem [1] we have that

Lemma 4.1. *[8, 22] Let $e = uv$ be a cut edge of a bipartite graph G, then*

$$b_{2k}(G) = b_{2k}(G-e) + b_{2k-2}(G-u-v).$$

Lemma 4.2. *If $n_1 \geq n_2 + 2$, then*

$$p_i(K_{n,s}^{n_1,n_2,\ldots,n_s}) \leq p_i(K_{n,s}^{n_1-1,n_2+1,\ldots,n_s}),$$

and the inequality is strict for some i.

Proof. It suffices to prove that $b_{2i}(S(K_{n,s}^{n_1,n_2,\ldots,n_s})) \leq b_{2i}(S(K_{n,s}^{n_1-1,n_2+1,\ldots,n_s}))$ for $0 \leq i \leq n$, since $b_{2i}(S(G)) = p_i(G)$ for $0 \leq i \leq n$. By Lemma 4.1, we have

$$b_{2i}(S(K_{n,s}^{n_1,n_2,\ldots,n_s})) = b_{2i}(S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) \cup P_2) + b_{2i-2}((S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) - v_1) \cup P_1)$$

$$b_{2i}(S(K_{n,s}^{n_1-1,n_2+1,\ldots,n_s})) = b_{2i}(S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) \cup P_2) + b_{2i-2}((S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) - v_2) \cup P_1)$$

It is easy to see that there exist cut edges $e_1, e_2, \ldots, e_{n_1 - n_2 - 1}$ such that

$$S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) - v_2 - e_1 - e_2 - \ldots - e_{n_1-n_2-1} \cong S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) - v_1.$$

By Lemma 4.1 we have that $b_{2i-2}((S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) - v_1) \cup P_1) \leq b_{2i-2}((S(K_{n,s}^{n_1-1,n_2,\ldots,n_s}) - v_2) \cup P_1)$ and it is strict for $i = 2$. Thus the result follows. $\square$

Lemma 4.3. *[11, 22] Let G be a simple graph, T be a tree with t edges, and $u \in V(G), v \in V(T)$. Let G_1 be the graph obtained from G and T by identifying the vertices u of G and v of T, G_2 be the graph obtained from G and the star S_{t+1} by identifying the vertex u of G and the unique central vertex of S_{t+1}. Then*

$$p_i(G_1) \geq p_i(G_2),$$

with equality for $0 \leq i \leq |V(G_1)|$ if and only if $T \cong S_{t+1}$ and v is its central vertex.

Theorem 4.1. *Let G be a simple connected graph on n vertices with clique number ω. Then*

$$IE(G) \geq IE(K_{n,\omega})$$

with equality if and only $G \cong K_{n,\omega}$.

Proof. By Lemma 2.3 and Lemma 4.3, there exist $n_1, n_2, \ldots, n_\omega$ such that $p_i(G) \geq p_i(K_{n,\omega}^{n_1,\ldots,n_\omega})$ for $0 \leq i \leq n$, the equalities hold if and only if $G \cong K_{n,\omega}^{n_1,\ldots,n_\omega}$. By Lemma 4.2 it follows that $p_i(K_{n,\omega}^{n_1,\ldots,n_\omega}) \geq p_i(K_{n,\omega})$ for $0 \leq i \leq n$, and the equalities hold if and only if $G \cong K_{n,\omega}$. Thus, $IE(G) \geq IE(K_{n,\omega})$ with equality if and only $G \cong K_{n,\omega}$. $\square$

5. The Incidence Energy of Connected Graphs with Given the Number of Pendent Vertex

Theorem 5.1. *Let G be a simple connected graph on n vertices with exactly k pendent vertices. Then*

$$IE(G) \leq IE(K_{n,n-k}^{n-k})$$

with equality if and only if $G \cong K_{n,n-k}^{n-k}$.

Proof. By Lemma 2.3 and Lemma 4.2, the result follows. $\square$

Lemma 5.1. *(i) [3, p. 53] Let $n = 4k$ or $4k + 2$. Then for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$,*

$$m(P_2 \cup P_{n-2}, i) \geq m(P_4 \cup P_{n-4}, i) \geq \ldots \geq m(P_{2k} \cup P_{n-2k}, i).$$

(ii) The above last inequality is strict for $i = 3$ for $n \geq 6$.

Proof. We only need to prove the second result. Note that for $a + b = n/2$,

$$
\begin{aligned}
m(P_{2a} \cup P_{2b}, 3) &= \binom{2a-3}{3} + \binom{2b-3}{3} + \binom{2a-2}{2}(2b-1) + \binom{2b-2}{2}(2a-1) \\
&= \frac{1}{6}n^3 - \frac{5}{2}n^2 + \frac{40}{3}n - 26 - abn,
\end{aligned}
$$

the (ii) is immediate for $n \geq 6$. $\square$

Lemma 5.2. *Let $G(k, l)$ be a connected graph obtained from a non-trivial connected graph G by attaching two pendent path of length l and k at the common vertex w of G. Let $k \geq l + 1$, then*
   (i)[11] $p_i(G(k, l)) \geq p_i(G(k - 1, l + 1))$ for $0 \leq i \leq |V(G(k, l))|$.
   (ii) If G is a tree with at least an edge, then the above inequality is strict for $i = 4$.

Proof. We will prove (*ii*). Since G is a tree, $S(G(k,l))$ is also a tree and

$$p_i(G(k,l)) = b_{2i}(S(G(k,l))) = m(S(G(k,l)),i).$$

Let $A_i(k,l)$ be the set of i-matching in $S(G(k,l))$, $B_i(k,l)$ be the set of i-matching in $S(G(k,l))$, each of which contains an edge of $S(G)$ adjacency to w, and $C_i(k,l) = A_i(k,l) \setminus B_i(k,l)$. Then

$$
\begin{aligned}
m(S(G(k,l)),i) &= |B_i(k,l)| + |C_i(k,l)| \\
m(S(G(k-1,l+1)),i) &= |B_i(k-1,l+1)| + |C_i(k-1,l+1)|
\end{aligned}
$$

Clearly, $|C_i(k,l)| = m(S(G) - w \cup P_{2k+2l+1}, i) = |C_i(k-1,l+1)|$. Let $N_{S(G)}(w) = \{w_1, w_2, \ldots, w_s\}$, then

$$
\begin{aligned}
|B_i(k,l)| &= \sum_{j=1}^{s} \sum_{t=1}^{i} m(S(G) - w - w_j, t-1) m(P_{2k} \cup P_{2l}, i-t) \\
&\geq \sum_{j=1}^{s} \sum_{t=1}^{i} m(S(G) - w - w_j, t-1) m(P_{2k-2} \cup P_{2l+2}, i-t) \\
&= |B_i(k-1,l+1)|,
\end{aligned}
$$

and the inequality is strict for $i = 4$ from Lemma 5.1. Thus $p_i(G(k,l)) \geq p_i(G(k-1,l+1))$ for $0 \leq i \leq n$, and the inequality is strict for $i = 4$. □

Lemma 5.3. *Let T be a tree on n vertices with exactly k pendent vertices. Let u be the vertex of degree not less than 3 in T such that there is a pendent path of length $a - 1$ pendent to u. Then*

$$IE(T) \geq IE(S_{n,k-1}^a)$$

with equality if and only if $T \cong S_{n,k-1}^a$.

Proof. Since $S(T)$ and $S(S_{n,k-1}^a)$ are trees on $2n - 1$ vertices, we only need to prove that for $1 \leq i \leq n$, $m(S(T),i) \geq m(S(S_{n,k-1}^a),i)$ with all equalities hold if and only if $T \cong S_{n,k-1}^a$. The result follows from the Appendix Table of [12] for $n \leq 10$. Suppose now that $n \geq 11$ and the result is true for the values less than n.

Let $N_{S(T)}(u) = \{u_1, u_2, \ldots, u_s\}$ and $\{v_1, v_2, \ldots, v_k\}$ be k pendent vertices in T, where $N_G(u)$ denotes the neighbor set of the vertex u in G. Suppose that $e_1 = uu_1$ be the edge of pendent path of length $2a - 2$ with pendent vertex v_1. Let $n - a = b(k-1) + r$, where $0 \leq r \leq k - 2$. By Lemma 4.1 we have

$$
\begin{aligned}
m(S(T),i) &= m(S(T) - e_1, i) + m(S(T) - u - u_1, i-1) \\
m(S(S_{n,k-1}^a),i) &= m(S(S_{n-a+1,k-1}) \cup P_{2a-2}, i) \\
&\quad + m(P_{2a-3} \cup rP_{2\lceil \frac{n-a}{k-1} \rceil} \cup (k-r-1)P_{2\lfloor \frac{n-a}{k-1} \rfloor}, i-1)
\end{aligned}
$$

Note that $S(T) - e_1 \cong S(T') \cup P_{2a-2}$, where T' is a tree on $n - a + 1$ vertices with $k - 1$ pendent vertices. If $k = 3$, then $S(T')$ is a path of $2n - 2a + 1$ vertices and $S(T) - e_1 \cong S(T') \cup P_{2a-2} \cong S(S_{n-a+1,2}) \cup P_{2a-2}$. Suppose that $k \geq 4$ and w is a vertex of degree greater than 2 in T' such that there is a pendent path of length b pendent to w. By induction hypothesis and Lemma 5.2, we have that for $0 \leq i \leq n$,

$$
\begin{aligned}
m(S(T) - e_1, i) &= m(S(T') \cup P_{2a-2}, i) \\
&\geq m(S(S_{n-a+1,k-2}^b) \cup P_{2a-2}, i) \\
&\geq m(S(S_{n-a+1,k-1}) \cup P_{2a-2}, i).
\end{aligned}
$$

In the following we will prove the inequality

$$m(S(T) - u - u_1, i-1) \geq m(P_{2a-3} \cup rP_{2\lceil \frac{n-a+1}{k-1} \rceil} \cup (k-r-1)P_{2\lfloor \frac{n-a+1}{k-1} \rfloor}, i-1)$$

for $1 \leq i \leq n$. Let $S(T) - u - u_1 \cong \bigcup_{i=1}^{s} G_i$, where $G_1 \cong P_{2a-3}$ and G_i is the component of $S(T) - u - u_1$ containing u_i for $2 \leq i \leq s$. Suppose that $V(G_2) \cap \{v_1, v_2, \ldots, v_k\} = \{v_2, \ldots, v_t\}$. Let P^2 be the unique path joining u_2 and

v_2. Let e_3 be the edge which is the closest to vertex v_3 and incident to a vertex of degree greater than 2 in G_2. And in process, for $4 \leq i \leq t$, let e_i be the edge which is the closest to vertex v_i and incident to a vertex of degree greater than 2 in $G_2 - \bigcup_{j=3}^{i-1} e_i$. Then $G_2 - e_3 - \ldots - e_t \cong \bigcup_{i=2}^{t} P^i$, where P^i is a path containing v_i for $3 \leq i \leq t$ in G_2. It is easy to see that P^i contains even vertex number. Repeating this procession to $G_3, \ldots, G_s$, we finally obtain a path decomposition of $S(T) - u - u_1$. Therefore $P_{2a-3} \bigcup_{j=2}^{k} P^j$ is a spanning subgraph of $S(T) - u - u_1$. Since the vertex number of all P^j ($2 \leq j \leq k$) are even, by Lemma 5.1 we have

$$
\begin{aligned}
m(S(T) - u - u_1, i - 1) &\geq m\left(P_{2a-3} \bigcup_{j=2}^{k} P^j, i - 1\right) \\
&\geq m\left(P_{2a-3} \cup rP_{2\lceil \frac{n-a}{k-1} \rceil} \cup (k - r - 1)P_{2\lfloor \frac{n-a}{k-1} \rfloor}, i - 1\right)
\end{aligned}
$$

the first equality for $i = 2$ if and only if $S(T)$ is a starlike tree with unique vertex u of degree greater than 3, and the second equality for $i = 3$ if and only if $\bigcup_{j=2}^{k} P^j \cong rP_{2\lceil \frac{n-a+1}{k-1} \rceil} \cup (k - r - 1)P_{2\lfloor \frac{n-a+1}{k-1} \rfloor}$. These imply all equalities hold if and only if $T \cong S_{n,k-1}^a$. $\square$

Theorem 5.2. *Let T be a tree on n vertices with k pendent vertices. Then*

$$IE(T) \geq IE(S_{n,k})$$

with equality if and only if $G \cong S_{n,k}$.

Proof. By Lemma 5.2 and Lemma 5.3, the result follows. $\square$

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